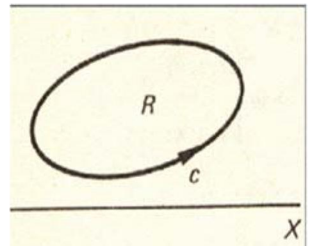
LECTURE NO 13&14

Green's Thorem

Let P and Q be two function of x and y that are finite and continuous inside and the boundary c of a region R in the xy-plane.If the first partial derivatives are continuous within the region and on the boundary, then Green's theorem states that.



— P_y — Q_x dx

dy = O_C (P dx+ Q dy)

R

That is, a double integral over the plane region R can be transformed into a line integral over the boundary c of the region and the action is reversible.
Let us see how it works.

EXAMPLE 4

Evaluate I = O_C {(2x y)dx + (2y+x)dy} around the boundary c of the ellipse $x^2 + 9y^2 = 16$.

Solution: The integral is of the form

I = O {P dx + Q dy) where P = 2x y — P_y = 1 and Q = 2y + x

— Q_x = 1.

C

I = — P_y — Q_x dxdy= R (1 1)dx dy= 2

R dx dy = 2A

R

But dx dy over any closed region give the area of the figure.

R

In this case, then, I = 24 where A is the area of the ellipse *A ab*

$x^2 + 9y^2 = 16$ i.e. $\frac{x^2}{16} + \frac{9y^2}{16} = 1$

a = 4; b = 3 A $\frac{4}{ab = 3}$ $\frac{16}{}$ I = 2A = $\frac{32}{3}$

To demonstrate the advantage of Green's theorem, let us work through the next example (a) by the previous method, and (b) by applying Green's theorem.

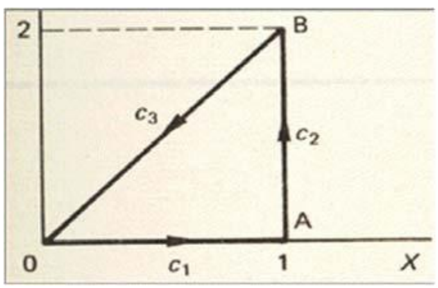
Example 5: Evaluate I = O {(2x+y) dx+(3x 2y) dy} taken in anticlockwise manner

C

round the triangle with vertices at O (0,0) A (1, 0) B (1, 2).

Solution: I = O {(2x + y) dx + (3x 2y) dy}

C



(a) By the previous method

There are clearly three stages with c_1,c_2,c_3. Work through the complete evaluation to determine the value of I. It will be good revision. When you have finished, check the result with the solution in the next frame. I = 2

(a) (i) c_1 is y = 0 dy = 0

1

1

I_1 = 2x dx = $\left[x^2\right]$ = 1 I_1 = 1

(ii) c_2 is x = 1 dx = 0

2

0

I_2= (3 2y) dy=$\left[3y \quad y^2\right]$ =2 I_2=2

0 1

(iii) c_3 is y = 2x dy = 2 dx

0

I_3 = {4x dx + (3x 4x) 2 dx}

1
0

$$= \quad 2x\,dx = \left[x^2\right]_1^0 = \quad 1 \qquad I_3 = \quad 1$$

$I = I_1+I_2+I_3 = 1+2+(\quad 1) = 2 \qquad I = 2$

Now we will do the same problem by applying Green's theorem, so more

(b) By Green's theorem

$$I = O_C \quad \{(2x + y)\,dx + (3x \quad 2y)\,dy\}$$

$$P = 2x + y \qquad \frac{P}{y} = 1;$$

$$Q = 3x \quad 2y \qquad \frac{Q}{x} = 3$$

$$I = {}_R \quad \frac{P}{y} \quad \frac{Q}{x} \quad dx\,dy$$

$$= {}_R \quad (1 \quad 3)\,dx\,dy = 2 \;{}_R \quad dx\,dy = 2A$$

$$= 2 \quad \text{the area of the triangle} = 2 \quad \frac{1}{2} \quad 1\;2 \quad = 2$$

$I = 2$

Remark: Application of Green's theorem is not always the quickest method. It is useful, however, to have both methods available.

If you have not already done so, make a note of Green's theorem.

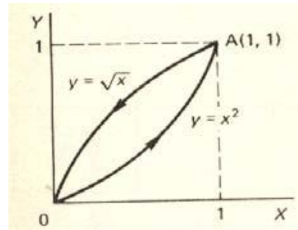
$$\frac{P}{y} \quad \frac{Q}{x} \quad dx\,dy = O_C \quad (P\,dx + Q\,dy)$$

R

Note: Green's theorem can, in fact, be applied to a region that is not simply connected by arranging a link between outer and inner boundaries, provided the path of integration is such that the region is kept on the left-hand side.

EXAMPLES

Example 1: Evaluate the line integral $I = O_C$ {xy dx + (2x y) dy} round the region bounded by the curves $y = x^2$ and $x = y^2$ by Green's theorem. Solution: Points of intersection are O(0, 0) and A(1, 1).

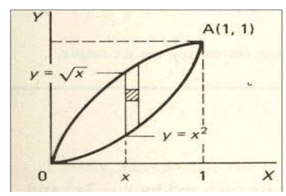


$$I = O_C \{xy\, dx + (2x \quad y)\, dy\}$$

$$= O_C \{Pdx + Qdy\} = {}_R \quad — \; \frac{P}{y} \quad — \; \frac{Q}{x} \quad dx\, dy$$

$$P = xy \quad \frac{P}{y} = x; \quad Q = 2x \quad y \quad \frac{Q}{x} = 2$$

$$I = {}_R (x \quad 2)\, dx\, dy = \int_0^1 \int_{y=x^2}^{y=\sqrt{x}} (x \quad 2)\, dy\, dx$$



$$= \int_0^1 (x \quad 2) \left[y \right]_{x^2}^{\sqrt{x}} dx$$

$$I = \int_0^1 (x \quad 2) \left(\sqrt{x} \quad x^2 \right) dx = \int_0^1 (x^{3/2} \quad x^3 \quad 2x^{1/2} + 2x^2)\, dx$$

$$= \left[\frac{2}{5} x^{5/2} \quad \frac{1}{4} x^4 \quad \frac{4}{3} x^{3/2} + \frac{2}{3} x^3 \right]_0^1 = \frac{31}{60}$$

In this special case when P=y and Q= x so $\frac{P}{y} = 1$ and $\frac{Q}{x} = \quad 1$

Green's theorem then states $\{1 \quad (\quad 1)\}\, dx\, dy = O \; (P\, dx + Q\, dy)$

R C

i.e. 2 dx dy = O (y dx x dy) = O (x dy y dx)
R C C

Therefore, the area of the closed region A = dx dy = $\frac{1}{2}$ O_C (x dy y dx)
R

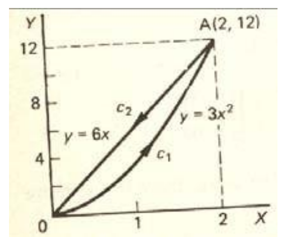
Example 2: Determine the area of the figure enclosed by $y = 3x^2$ and $y = 6x$.

Solution: Points of intersection: $3x^2 = 6x$ $x = 0$ or 2

Area A = $\frac{1}{2}$ O_C (x dy y dx) We evaluate the integral in two parts,

i.e.

OA along c_1 and AO along c_2



2A=$_{c_1}$ ((along OA xdy ydx)) +$_{c_2}$ ((along OA xdy ydx)) = I_1+I_2

I_1: c_1 is $y = 3x^2$ dy = 6x dx

I_1 = (6x²dx 3x²dx) = 3x² dx = $[x^3]$ = 8 $I_1 = 8$
2 2 2
0 0 0

Similarly, for c_2 is y = 6x dy = 6 dx
0

I_2 = (6x dx 6x dx) = 0
2

$I_2 = 0$

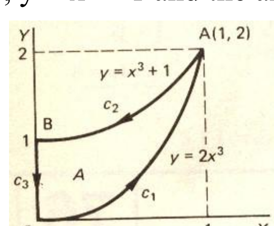
$I = I_1 + I_2 = 8 + 0 = 8$

A = 4 square units

Example 3: Determine the area bounded by the curves $y = 2x^3$, $y = x^3 + 1$ and the axis $x = 0$ for $x \geq 0$.

Solution: Here it is $y = 2x^3$; $y = x^3 + 1$; $x = 0$

Point of intersection $2x^3 = x^3 + 1$ $x^3 = 1$ $x = 1$



$$\text{Area } A = \frac{1}{2}\oint_C (x\,dy \quad y\,dx) \qquad 2A = \oint_C (x\,dy \quad y\,dx)$$

(a) OA : c_1 is $y = 2x^3$ $\quad dy = 6x^2\,dx$

$$I_1 = {}_{c_1}(xdy \quad ydx) = {}_0^1(6x^3dx \quad 2x^3\,dx) \quad = \quad {}_0^1 4x3\,dx = \Big[\,x^4\,\Big]_0^1 = 1$$

$I_1 = 1$

(b) AB: c_2 is $y = x^3 + 1$ $\quad dy = 3x^2\,dx$

$$I_2 = {}_1^0\{3x^3\,dx \quad (x^3 + 1)\,dx\} = {}_1^0(2x^3 \quad 1)\,dx \; = \; 2\;\frac{x}{4}\;{}_1^0 = \;\frac{1}{2}\quad 1 = \quad 2\frac{1}{}$$

$I_2 = \frac{1}{2}$

(c) BO: c_3 is $x = 0$ $\quad dx = 0$

$$I_3 = {}_{y=1}^{y=0}(xdy \quad ydx) = 0 \qquad I_3 = 0$$

$$2A = I = I_1 + I_2 + I_3 = 1 + \frac{1}{2} + 0 = 1\frac{1}{2} \qquad A = \frac{3}{4} \text{ square units}$$

- **Exact differential**

  If P dx + Q dy is an exact differential, then

  (a) $\dfrac{P}{y} = \dfrac{Q}{x}$

  (b) $I = {}_c(P\,dx + Q\,dy)$ is independent of the path of integration

  (c) $I = \oint_C (P\,dx + Q\,dy)$ is zero.

- **Exact differential in three variables.**

  If P dx + Q dy + R dw is an exact differential

  $\quad P \quad Q \qquad P \quad R \qquad R \quad Q$

(a) $=\;;\quad=\;;\quad=$

$y \quad x \quad w \quad x \quad y \quad w$

(b) $_c$ (P dx + Q dy + R dw) is independent of the path of integration.

(c) $\underset{C}{O}$ (P dx + Q dy + R dw) is zero.

- **Green's theorem**

$$O_C \, (P\,dx+Q\,dy)= {}_R \quad — \; \underline{P}_y \quad — \; \underline{Q}_x \quad dx\,dy$$ and,

for a simple closed curve,

$$\underset{C}{O} \, (x\,dy \quad y\,dx) = 2 \underset{R}{} \; dx\,dy = 2A$$

where A is the area of the enclosed figure.